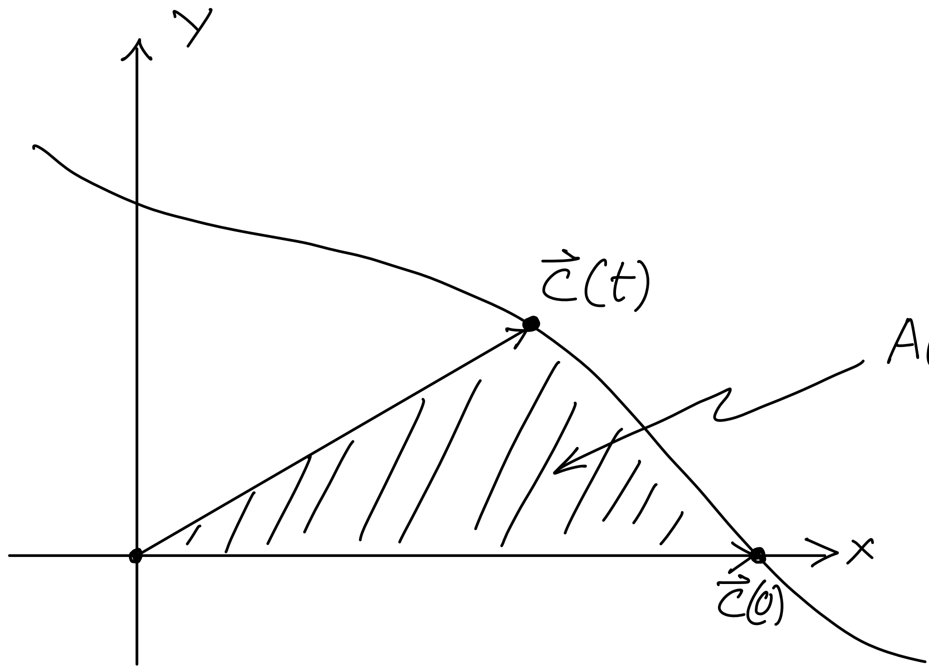
§6. The Kepler Problem

Consider an object moving along a trajectory $\vec{c}(t)$, where t is time:



We want to compute the area swept out after a time t.

To this end, it is useful to switch to polar coordinates where we have:

$$\vec{c}(t) = r(t)\left(\cos\theta(t), \sin(\theta)\right)$$

The curve $\vec{c}(t)$, $0 \leq t \leq T$ is then called a "parametric curve".

We introduce the notation:

$$\vec{c}(t) = r(t) \cdot \vec{e}(\theta(t)) \qquad (1)$$

where

$$\vec{e}(t) = (\cos t, \sin t)$$

is just the parametrized curve that runs along the unit circle.
Note that

$$\vec{e}\,'(t) = (-\sin t, \cos t)$$

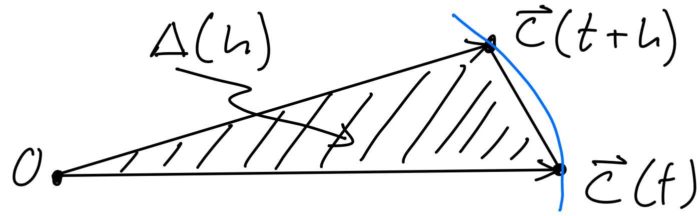
is also a unit vector but perpendicular to $\vec{e}(t)$.

$$\Longrightarrow \quad \det(\vec{e}(t), \vec{e}\,'(t)) = 1 \qquad (2)$$

explicitly:

$$\det\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} = \cos^2 t - (-\sin^2 t) = 1$$

In order to compute the area $A(t)$, it's useful to first obtain a formula for $A'(t)$ $\longrightarrow$ consider triangle:



The area of such a triangle can be expressed as:

$$\text{area}(\Delta(t)) = \frac{1}{2} \det\left(\vec{c}(t), \vec{c}(t+h) - \vec{c}(t)\right)$$

Then we compute

$$\begin{aligned} A'(t) &= \lim_{h \to 0} \frac{A(t+h) - A(t)}{h} \\ &= \lim_{h \to 0} \frac{\text{area}(\Delta(h))}{h} \\ &= \frac{1}{2} \det\left(\vec{c}(t), \lim_{h \to 0} \frac{\vec{c}(t+h) - \vec{c}(t)}{h}\right) \\ &= \frac{1}{2} \det\left(\vec{c}(t), \vec{c}\,'(t)\right) \end{aligned}$$

Using

$$\vec{c}\,'(t) = r'(t) \cdot \vec{e}(\theta(t)) + r(t)\, \theta'(t)\, \vec{e}\,'(\theta(t)) \quad (3)$$

and linearity of the determinant in its column vectors, we obtain:

$$\begin{aligned} \det\left(\vec{c}(t), \vec{c}\,'(t)\right) &= r(t) r'(t) \underbrace{\det\left(\vec{e}(\theta(t)), \vec{e}(\theta(t))\right)}_{=0} \\ &\quad + r(t)^2 \theta'(t) \underbrace{\det\left(\vec{e}(\theta(t)), \vec{e}\,'(\theta(t))\right)}_{=1} \\ &= r^2 \theta' \qquad (4) \end{aligned}$$

$\longrightarrow$ total area can be obtained by integration:

$$A(t) = \frac{1}{2} \int_0^t r(t)^2 \theta'(t)\, dt$$

or, performing the substitution $d\phi = \Theta'(t)dt$:

$$A(\theta) = \frac{1}{2}\int_0^{\theta(t)} \rho(\phi)^2 \, d\phi$$

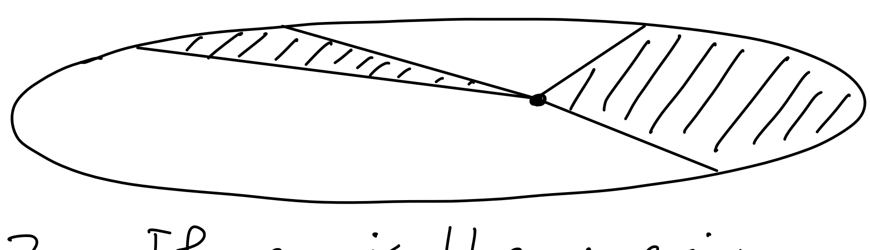
where $\rho = r \circ \Theta^{-1}$. Check:

$$A'(t) = \frac{1}{2}\rho(\theta(t))^2 \cdot \theta'(t)$$
$$= \frac{1}{2} r(t)^2 \theta'(t) \qquad (5)$$

Let us now use these results to understand Kepler's 3 laws of planetary motion (1609):

K1: The planets move in ellipses, with the sun at one focus.

K2: Equal areas are swept out by the radius vector in equal times



K3: If a is the major axis of a planet's elliptical orbit and T its period, then a^3/T^2 is the same for all planets

In our notation, Kepler's second law is equivalent to saying that $A'(t)$ is constant.

so $K2 \Longleftrightarrow A'' = 0$

But

$$A'' = \frac{1}{2}\left[\det(\vec{c}, \vec{c}\,')\right]' = \frac{1}{2}\underbrace{\det(c', c')}_{=0} + \frac{1}{2}\det(c, c'')$$

$$= \frac{1}{2}\det(\vec{c}, \vec{c}\,'')$$

(Homework)

So

$$K2 \Longleftrightarrow \det(\vec{c}, \vec{c}\,'') = 0$$

From this we can deduce the following:

<u>Proposition 1</u> (Newton):

Kepler's second law is true if and only if there exists a force $\vec{F}$ that is central, and in this case each planetary path $\vec{c}(t) = r(t) \cdot \vec{e}(\theta(t))$ satisfies the equation

$$r^2\theta' = \det(\vec{c}, \vec{c}\,') = \text{const.} \quad (K2)$$

<u>Proof</u>:

Know that $\vec{F} = m \cdot \vec{c}\,''(t)$

Saying that the force is central just means that it always points along $\vec{c}$
Since $\vec{c}\,''(t)$ is in the direction of the force, that is equivalent to saying that $\vec{c}\,''(t)$ always points along $\vec{c}(t)$

$$\Rightarrow \det\left(\vec{c}(t), \vec{c}\,''(t)\right) = 0$$

Thus $\det\left(\vec{c}(t), \vec{c}\,'(t)\right) = \text{constant}$ $\square$

We are now in the position to derive Kepler's first law from Newton's concept of a "gravitational force":

<u>Proposition 2</u> (Newton):
If the gravitational force of the sun is a central force that satisfies an "inverse square law", then the path of any planet in it will be an ellipsis having the sun at one focus.

<u>Proof</u>:

By K2 we have $r^2\theta' = \det(\vec{c}, \vec{c}\,') = \widetilde{M}$

for some constant $\tilde{M}$. The hypothesis of an inverse square law can be written as

$$\vec{c}\,''(t) = -\frac{H}{r(t)^2}\,\vec{e}(\theta(t))$$

for some constant H. Using K2, this can be written as

$$\frac{\vec{c}\,''(t)}{\theta'(t)} = -\frac{H}{\tilde{M}}\,\vec{e}(\theta(t))$$

Notice that the left-hand side of this equation is

$$[\vec{c}\,'(t)\circ\theta^{-1}]'(\theta(t))$$ (use inverse function derivative and chain rule)

So if we let

$$D = \vec{c}\,'\circ\theta^{-1},$$

then the equation can be written as

$$D'(\theta) = -\frac{H}{\tilde{M}}\,\vec{e}(\theta) = -\frac{H}{\tilde{M}}(\cos\theta, \sin\theta),$$

where we now view θ as an independent variable. Integrating gives

$$D(\theta) = \left(\frac{H\cdot\sin\theta}{-\tilde{M}} + A,\ \frac{H\cdot\cos\theta}{\tilde{M}} + B\right)$$

for two constants A and B.
Reintroducing the dependence on t, we have

$$\vec{c}\,'(t) = \left(\frac{H \cdot \sin\theta(t)}{-\tilde{M}} + A, \ \frac{H \cdot \cos\theta(t)}{\tilde{M}} + B \right)$$

Substituting this together with $\vec{c} = r(\cos\theta, \sin\theta)$, into the equation

$$\det(\vec{c}, \vec{c}\,') = \tilde{M},$$

we get

$$r\left[\frac{H}{\tilde{M}} \cos^2\theta + B\cos\theta + \frac{H}{\tilde{M}} \sin^2\theta - A\sin\theta \right] = \tilde{M},$$

which simplifies to

$$r\left[\frac{H}{\tilde{M}^2} + \frac{B}{\tilde{M}} \cos\theta - \frac{A}{\tilde{M}} \sin\theta \right] = 1.$$

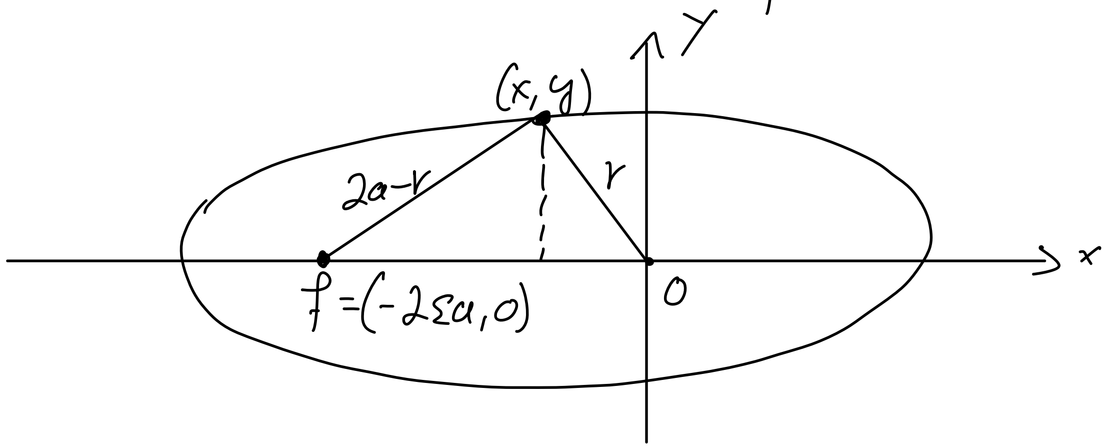
This can be rewritten as (Homework):

$$r(t)\left[\frac{H}{\tilde{M}^2} + C\cos\big(\theta(t) + D\big) \right] = 1,$$

for some constants C and D. By choosing our polar axis appropriately (which ray corresponds to $\theta = 0$), we can let $D = 0$

$$\Rightarrow r\left[1 + \varepsilon \cos\theta \right] = \frac{\tilde{M}^2}{H} =: \Lambda$$

But this is the formula for an ellipsis which can be seen as follows:



with $0 \leq \varepsilon < 1$. We have

$$r^2 = x^2 + y^2 \qquad (1)$$

$$\Rightarrow (2a-r)^2 = (x-(-2\varepsilon a))^2 + y^2,$$

$$\text{or } 4a^2 - 4ar + r^2 = x^2 + 4\varepsilon a x + 4\varepsilon^2 a^2 + y^2 \qquad (2)$$

Subtracting (1) by (2) and dividing by $4a$, we get

$$a - r = \varepsilon x + \varepsilon^2 a$$

$$\Leftrightarrow \quad r = a - \varepsilon x - \varepsilon^2 a = (1-\varepsilon^2)a - \varepsilon x$$

$$\Leftrightarrow \quad r = \Lambda - \varepsilon x, \text{ for } \Lambda = (1-\varepsilon^2)a$$

Using $x = r\cos\theta$, we have finally:

$$r(1 + \varepsilon\cos\theta) = \Lambda. \qquad \square$$

Let us see what this means for our original equation

$$\vec{c}\,''(t) = -\frac{H}{r(t)^2}\,\vec{e}(\theta(t))$$

$$= -\frac{\tilde{M}^2}{\Lambda}\cdot\frac{1}{r^2}\,\vec{e}(\theta(t)) \qquad (*)$$

We also see that the major axis of our ellipsis is given by:

$$a = \frac{\Lambda}{1-\varepsilon^2} \qquad (3)$$

while the minor axis is given by:

$$b = \frac{\Lambda}{\sqrt{1-\varepsilon^2}} \qquad (4)$$

$$\Longrightarrow \quad \frac{b^2}{\Lambda} = a \qquad (5)$$

Recall that

$$A'(t) = \frac{1}{2}r^2\theta' = \frac{1}{2}\tilde{M}$$

$$\longrightarrow \quad A(t) = \frac{1}{2}\tilde{M}\,t$$

$\longrightarrow$ area of the ellipse $= A(T) = \frac{1}{2}\tilde{M}\underset{\substack{\text{period}\\\text{of planet}}}{T}$

$$\Longleftrightarrow \quad \tilde{M} = \frac{2\,(\text{area of ellipse})}{T} = \frac{2\pi ab}{T}$$

Hence the constant $\tilde{M}^2/\Lambda$ in (*) is

$$\frac{\tilde{M}^2}{\Lambda} = \frac{4\pi^2 a^2 b^2}{T^2 \Lambda} \overset{(5)}{=} \frac{4\pi^2 a^3}{T^2}$$

From this we obtain

<u>Proposition 3</u> (Newton):

Kepler's third law is true if and only if the acceleration $\vec{c}\,''(t)$ of any planet, moving on an ellipse, satisfies

$$\vec{c}\,''(t) = -\tilde{G} \cdot \frac{1}{r^2}\, \vec{e}(\theta(t))$$

for a constant $\tilde{G}$ independent of the planet.

But what is the constant $\tilde{G}$?

Let us look at the force exercised on the planet due to the sun!

$$\vec{F} = m\vec{c}\,''(t) = -\frac{m\tilde{G}}{r^2}\, \vec{e}(\theta(t))$$

where m is the mass of the planet

But: Due to Newton's 3rd law, the planet exercises a force of the <u>same</u> magnitude on the sun (imagine the planet being much larger than the sun $\rightarrow$ sun would orbit around it!)

$\rightarrow$ $\tilde{G}$ must be proportional to M, i.e. $\tilde{G} = G \cdot M$, where M is the mass of the sun

The constant G is called Newton's constant is given by

$$G = 6.67 \cdot 10^{-11}\ N m^2 kg^{-2}$$

and the "universal gravitational law" takes the form:

$$\vec{F}_{grav} = -\frac{G m M}{r^2}\, \vec{e}(\theta(t))$$